

T-duality as coordinates permutation in double space *

B. Sazdović [†]

*Institute of Physics,
University of Belgrade,
11001 Belgrade, P.O.Box 57, Serbia*

December 6, 2016

Abstract

We introduce the $2D$ dimensional double space with the coordinates $Z^M = (x^\mu, y_\mu)$ which components are the coordinates of initial space x^μ and its T-dual y_μ . We shall show that in this extended space the T-duality transformations can be realized simply by exchanging places of some coordinates x^a , along which we want to perform T-duality and the corresponding dual coordinates y_a . In such approach it is evident that T-duality leads to the physically equivalent theory and that complete set of T-duality transformations form subgroup of the $2D$ permutation group. So, in the double space we are able to represent the backgrounds of all T-dual theories in unified manner.

1 Introduction

T-duality of the closed string has been investigated for a long time [1, 2, 3, 4]. It transforms the theory of a string moving in a toroidal background into the theory of a string moving in different toroidal background. Generally, one suppose that background has some continuous isometries which leaves the action invariant. In suitable adopted coordinates, where the isometry acts as translation, it means that background does not depend on some coordinates.

In the paper [5] the new procedure for T-duality of the closed string, moving in D dimensional weakly curved space, has been considered. The generalized approach allows one to perform T-duality along coordinates on which the Kalb-Ramond field depends. In that article T-duality transformations has been performed simultaneously along all coordinates. It corresponds to $T^{full} = T^0 \circ T^1 \circ \dots \circ T^{D-1}$ -duality relation with transformation

*Work supported in part by the Serbian Ministry of Education and Science, under contract No. 171031.

[†]e-mail: sazdovic@ipb.ac.rs

of the coordinates $y_\mu = y_\mu(x^\mu)$ connecting the beginning and the end of the T-duality chain

$$\Pi_{\pm\mu\nu}, x^\mu \xrightarrow{T_1} \Pi_{1\pm\mu\nu}, x_1^\mu \xrightarrow{T_2} \Pi_{2\pm\mu\nu}, x_2^\mu \xrightarrow{T_3} \dots \xrightarrow{T_D} \Pi_{D\pm\mu\nu} = {}^*\Pi_{\pm\mu\nu}, x_D^\mu = y_\mu. \quad (1.1)$$

Here $\Pi_{i\pm\mu\nu}$ and x_i^μ , ($i = 1, 2, \dots, D$) are background fields and the coordinates of the corresponding configurations. Applying the proposed procedure $T_{full} = T_0 \circ T_1 \circ \dots \circ T_{D-1}$ to the T-dual theory one can obtain the initial theory and the inverse duality relation $x^\mu = x^\mu(y_\mu)$, connecting the end and the beginning of the T-duality chain. For simplicity, in the article [5] T-duality has been performed along all directions. The nontrivial extension of this approach, compared with the flat space case, is a source of closed string non-commutativity [6, 7, 8].

In D -dimensional space it is possible to perform T-duality along any subset of coordinates $x^a : T^a = T^0 \circ T^1 \circ \dots \circ T^{d-1}$, and along corresponding T-dual ones $y_a : T_a = T_0 \circ T_1 \circ \dots \circ T_{d-1}$, ($a = 0, 1, \dots, d-1$). In the paper [9] this was done for the string moving in the weakly curved background. For each case the T-dual actions, T-dual background fields and T-duality transformations has been obtained. Let us stress that T-dualization $\mathcal{T}^a = T^a \circ T_a$ of the present paper in the $2D$ dimensional double space contains two T-dualizations in terminology of Ref.[9]. In fact D dimensional T-dualizations T^a and T_a of the present paper are denoted \mathcal{T}^a and \mathcal{T}_a in Ref.[9].

The introduction of the extended space of double dimensions with coordinates $Z^M = (x^\mu, y_\mu)$ will help us to reproduce all results of Ref.[9] and offer simple explanation for T-duality. In the present article we will demonstrate this for the flat background, while for the weakly curved background it will be presented elsewhere [10]. For example, T-duality T^{μ_1} (along fixed coordinate x^{μ_1}) and T-duality T_{μ_1} (along corresponding dual coordinate y_{μ_1}) can be performed simply by exchanging the places of the coordinates x^{μ_1} and y_{μ_1} in the double space. It can be realized just multiplying Z^M with constant $2D \times 2D$ matrix. Similarly, arbitrary T-duality $\mathcal{T}^a = T^a \circ T_a$ can be realized by exchanging the places of the coordinates $x^{\mu_1}, x^{\mu_2}, \dots, x^{\mu_{d-1}}$ with the corresponding dual coordinates $y_{\mu_1}, y_{\mu_2}, \dots, y_{\mu_{d-1}}$. From this explanation it is clear that T-duality leads to the equivalent theory, because permutation of the coordinates in double space can not change the physics.

Similar approach to T-duality, as a transformation in double space, appeared long time ago [11]-[15]. Interest in this topic emerged again with the articles [16, 17]. In the paper [11] the beginning and the end of the chain (1.1) has been established. The relation of our approach and Ref.[16] will be discussed in Sec.4.

The basic tools in our approach are T-duality transformations connected beginning and end of the chain. Rewriting these transformations in the double space we obtain the fundamental expression, where the generalized metric relate derivatives of the extended coordinates. We will show that this expression is enough to find background fields from

every nodes of the chain and T-duality transformations between arbitrary nodes. In such a way we unify the beginning and all corresponding T-dual theories of the chain (1.1).

2 T-duality in the double space

Let us consider the closed bosonic string which propagates in D-dimensional space-time described by the action [18]

$$S[x] = \kappa \int_{\Sigma} d^2\xi \sqrt{-g} \left[\frac{1}{2} g^{\alpha\beta} G_{\mu\nu}[x] + \frac{\epsilon^{\alpha\beta}}{\sqrt{-g}} B_{\mu\nu}[x] \right] \partial_{\alpha} x^{\mu} \partial_{\beta} x^{\nu}, \quad (\epsilon^{01} = -1). \quad (2.1)$$

The string, with coordinates $x^{\mu}(\xi)$, $\mu = 0, 1, \dots, D-1$ is moving in the non-trivial background, defined by the space-time metric $G_{\mu\nu}$ and the Kalb-Ramond field $B_{\mu\nu}$. Here $g_{\alpha\beta}$ is intrinsic world-sheet metric and the integration goes over two-dimensional world-sheet Σ with coordinates ξ^{α} ($\xi^0 = \tau$, $\xi^1 = \sigma$).

The requirement of the world-sheet conformal invariance on the quantum level leads to the space-time equations of motion, which in the lowest order in slope parameter α' , for the constant dilaton field $\Phi = \text{const}$ are

$$R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_{\nu}{}^{\rho\sigma} = 0, \quad D_{\rho} B^{\rho}{}_{\mu\nu} = 0. \quad (2.2)$$

Here $B_{\mu\nu\rho} = \partial_{\mu} B_{\nu\rho} + \partial_{\nu} B_{\rho\mu} + \partial_{\rho} B_{\mu\nu}$ is the field strength of the field $B_{\mu\nu}$, and $R_{\mu\nu}$ and D_{μ} are Ricci tensor and covariant derivative with respect to space-time metric.

We will consider the simplest solutions of (2.2)

$$G_{\mu\nu} = \text{const}, \quad B_{\mu\nu} = \text{const}, \quad (2.3)$$

which satisfies the space-time equations of motion.

Choosing the conformal gauge $g_{\alpha\beta} = e^{2F} \eta_{\alpha\beta}$, and introducing light-cone coordinates $\xi^{\pm} = \frac{1}{2}(\tau \pm \sigma)$, $\partial_{\pm} = \partial_{\tau} \pm \partial_{\sigma}$, the action (2.1) can be rewritten in the form

$$S = \kappa \int_{\Sigma} d^2\xi \partial_{+} x^{\mu} \Pi_{+\mu\nu} \partial_{-} x^{\nu}, \quad (2.4)$$

where

$$\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2} G_{\mu\nu}. \quad (2.5)$$

2.1 Standard sigma-model T-duality

Applying the T-dualization procedure on all the coordinates, we obtain the T-dual action [5]

$$S[y] = \kappa \int d^2\xi \partial_{+} y_{\mu} {}^* \Pi_{+}^{\mu\nu} \partial_{-} y_{\nu} = \frac{\kappa^2}{2} \int d^2\xi \partial_{+} y_{\mu} \theta_{-}^{\mu\nu} \partial_{-} y_{\nu}, \quad (2.6)$$

where

$$\theta_{\pm}^{\mu\nu} \equiv -\frac{2}{\kappa}(G_E^{-1}\Pi_{\pm}G^{-1})^{\mu\nu} = \theta^{\mu\nu} \mp \frac{1}{\kappa}(G_E^{-1})^{\mu\nu}. \quad (2.7)$$

Here we consider flat background and omit argument dependence of. Ref. [5]. The symmetric and antisymmetric parts of $\theta_{\pm}^{\mu\nu}$ are the inverse of the effective metric $G_{\mu\nu}^E$ and the non-commutativity parameter $\theta^{\mu\nu}$

$$G_{\mu\nu}^E \equiv G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}, \quad \theta^{\mu\nu} \equiv -\frac{2}{\kappa}(G_E^{-1}BG^{-1})^{\mu\nu}. \quad (2.8)$$

Consequently, the T-dual background fields are

$${}^*G^{\mu\nu} = (G_E^{-1})^{\mu\nu}, \quad {}^*B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu}. \quad (2.9)$$

Note that the dual effective metric is just inverse of the initial one

$${}^*G_E^{\mu\nu} \equiv {}^*G^{\mu\nu} - 4({}^*B{}^*G^{-1}{}^*B)^{\mu\nu} = (G^{-1})^{\mu\nu}, \quad (2.10)$$

and the following relations valid

$$({}^*B{}^*G^{-1})^{\mu}{}_{\nu} = -(G^{-1}B)^{\mu}{}_{\nu}, \quad ({}^*G^{-1}{}^*B)_{\mu}{}^{\nu} = -(BG^{-1})_{\mu}{}^{\nu}. \quad (2.11)$$

2.2 T-duality transformations

The T-duality transformations between all initial coordinates x^{μ} and all dual coordinates y_{μ} of the closed string theory have been derived in ref.[5]

$$\partial_{\pm}x^{\mu} \cong -\kappa\theta_{\pm}^{\mu\nu}\partial_{\pm}y_{\nu}, \quad \partial_{\pm}y_{\mu} \cong -2\Pi_{\mp\mu\nu}\partial_{\pm}x^{\nu}. \quad (2.12)$$

They are inverse to one another. We omit argument dependence and β_{μ}^{\pm} functions because they appear only in the weakly curved background.

We can put above T-duality transformations in a useful form, where on the left hand side we put the terms with world-sheet antisymmetric tensor $\varepsilon_{\alpha}^{\beta}$ (note that $\varepsilon_{\pm}^{\pm} = \pm 1$)

$$\begin{aligned} \pm\partial_{\pm}y_{\mu} &\cong G_{\mu\nu}^E\partial_{\pm}x^{\nu} - 2[BG^{-1}]_{\mu}{}^{\nu}\partial_{\pm}y_{\nu}, \\ \pm\partial_{\pm}x^{\mu} &\cong 2[G^{-1}B]^{\mu}{}_{\nu}\partial_{\pm}x^{\nu} + (G^{-1})^{\mu\nu}\partial_{\pm}y_{\nu}. \end{aligned} \quad (2.13)$$

Let us introduce the $2D$ dimensions double target space, which will play important role in the present article. It contains both initial and T-dual coordinates

$$Z^M = \begin{pmatrix} x^{\mu} \\ y_{\mu} \end{pmatrix}. \quad (2.14)$$

Here, as well as in Double field theory (for recent reviews see [19]-[22]), all coordinates are doubled. It differs from approach of Ref.[16] where only coordinates on the torus along

which we perform T-dualization are doubled. The relation of our and that of Ref.[16] will be discussed in Sec.4.

In terms of double space coordinate we can rewrite the T-duality relations (2.13) in the simple form

$$\partial_{\pm} Z^M \cong \pm \Omega^{MN} \mathcal{H}_{NK} \partial_{\pm} Z^K, \quad (2.15)$$

where

$$\Omega^{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (2.16)$$

is a constant symmetric matrix and we introduced so called generalized metric as

$$\mathcal{H}_{MN} = \begin{pmatrix} G_{\mu\nu}^E & -2 B_{\mu\rho} (G^{-1})^{\rho\nu} \\ 2(G^{-1})^{\mu\rho} B_{\rho\nu} & (G^{-1})^{\mu\nu} \end{pmatrix}. \quad (2.17)$$

It is easy to check that

$$\mathcal{H}^T \Omega \mathcal{H} = \Omega. \quad (2.18)$$

As noticed in Ref.[11], the relation (2.18) shows that there exists manifest $O(D, D)$ symmetry. In Double field theory it is usual to call Ω^{MN} the $O(D, D)$ invariant metric and denote with η^{MN} .

2.3 Equations of motions as consistency condition of T-duality relations

It is well known that the equation of motion and the Bianchi identity of the original theory are equal to the Bianchi identity and the equation of motion of the T-dual theory [11, 23, 5, 7]. The consistency conditions of the relations (2.15)

$$\partial_+ [\mathcal{H}_{MN} \partial_- Z^N] + \partial_- [\mathcal{H}_{MN} \partial_+ Z^N] \cong 0, \quad (2.19)$$

in components take a form

$$\partial_+ \partial_- x^\mu \cong 0, \quad \partial_+ \partial_- y_\nu \cong 0. \quad (2.20)$$

They are the equations of motion for both initial and T-dual theories.

The expression (2.19) originated from conservation of the topological currents $i^{\alpha M} = \varepsilon^{\alpha\beta} \partial_\beta Z^M$. It is often called Bianchi identity. In this sense T-duality in the double space unites equations of motion and Bianchi identities in a single relation (2.19) as is shown in [11].

We can write the action

$$S = \frac{\kappa}{4} \int d^2 \xi \partial_+ Z^M \mathcal{H}_{MN} \partial_- Z^N, \quad (2.21)$$

which variation produce the eq.(2.19).

3 T-duality as coordinates permutations in double space

Let us mark the T-dualization along some direction x^{μ_1} by T^{μ_1} , and its inverse along corresponding direction y_{μ_1} by T_{μ_1} . Up to now we collected the results from T-dualizations along all directions x^μ ($\mu = 0, 1, \dots, D-1$), $T^{full} = T^0 \circ T^1 \circ \dots \circ T^{D-1}$ and from its inverse along all directions y_μ $T_{full} = T_0 \circ T_1 \circ \dots \circ T_{D-1}$. So, the relation (2.15) in fact contains T-dualizations along all directions x^μ and y_μ $\mathcal{T} = T^{full} \circ T_{full}$.

In this section we will show that relation (2.15) contains information about any individual T-dualizations along some direction x^{μ_1} and corresponding one y_{μ_1} for fixed μ_1 ($\mathcal{T}^{\mu_1} = T^{\mu_1} \circ T_{\mu_1}$). Applying the same procedure to the arbitrary subset of directions we will be able to obtain all possible T-dualizations. It means that we are able to connect any two backgrounds in the chain (1.1) and treat all theories connected by T-dualities in a unified manner.

Let us split coordinate index μ into a and i ($a = 0, \dots, d-1$, $i = d, \dots, D-1$), and perform T-dualization along direction x^a and y_a

$$\mathcal{T}^a = T^a \circ T_a, \quad T^a \equiv T^0 \circ T^1 \circ \dots \circ T^{d-1}, \quad T_a \equiv T_0 \circ T_1 \circ \dots \circ T_{d-1}. \quad (3.1)$$

We will show that such T-dualization can be obtained just by exchanging places of coordinates x^a and y_a . Note that the double space contains coordinates of two spaces which are totally dual relative to one another. In the beginning these two theories are the initial one $S(x^\mu)$ and its T-dual along all coordinates $S(y_\mu)$. Arbitrary T-dualization in the double space along d coordinate with index a , \mathcal{T}^a , transforms at the same time $S(x^\mu)$ to $S[y_a, x^i]$ and $S(y_\mu)$ to $S[x^a, y_i]$. The obtained theories are also totally T-dual relative to one another.

3.1 The coordinates permutations in double space

Permutation of the initial coordinates x^a with its T-dual y_a we can realize by multiplying double space coordinate (2.14), now written as

$$Z^M = \begin{pmatrix} x^a \\ x^i \\ y_a \\ y_i \end{pmatrix}, \quad (3.2)$$

by the constant symmetric matrix $(\mathcal{T}^a)^T = \mathcal{T}^a$

$$\mathcal{T}^a{}^M{}_N = \begin{pmatrix} 1 - P_a & P_a \\ P_a & 1 - P_a \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1_a & 0 \\ 0 & 1_i & 0 & 0 \\ 1_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_i \end{pmatrix}. \quad (3.3)$$

Here P_a is $D \times D$ projector with d units on the main diagonal

$$P_a = \begin{pmatrix} 1_a & 0 \\ 0 & 0 \end{pmatrix}, \quad (3.4)$$

where 1_a and 1_i are d and $D - d$ dimensional identity matrices. In Ref.[3] this transformation is called factorized duality.

Note also that

$$(\mathcal{T}^a \mathcal{T}^a)^M{}_N = \delta^M{}_N, \quad (\Omega \mathcal{T}^a \Omega)^M{}_N = (\mathcal{T}^a)^M{}_N, \quad \mathcal{T}^a \Omega \mathcal{T}^a = \Omega. \quad (3.5)$$

The last relation means that $\mathcal{T}^a \in SO(D, D)$. More precisely, we will see that \mathcal{T}^a is in fact element of permutation group, which is a subgroup of $SO(D, D)$.

We will require that the dual extended space coordinate,

$$Z_a^M = \mathcal{T}^a{}^M{}_N Z^N = \begin{pmatrix} y_a \\ x^i \\ x^a \\ y_i \end{pmatrix}, \quad (3.6)$$

satisfy the same form of the T-duality transformations (2.15) as the initial one

$$\partial_{\pm} Z_a^M \cong \pm \Omega^{MN}{}_a \mathcal{H}_{NK} \partial_{\pm} Z_a^K. \quad (3.7)$$

Consequently, with the help of second equation (3.5) we find the dual generalized metric

$${}_a \mathcal{H} = \mathcal{T}^a \mathcal{H} \mathcal{T}^a, \quad (3.8)$$

or explicitly

$${}_a \mathcal{H}_{MN} = \begin{pmatrix} (G^{-1})^{ab} & 2(G^{-1}b)^a{}_j & 2(G^{-1}b)^a{}_b & (G^{-1})^{aj} \\ -2(bG^{-1})^b{}_i & g_{ij} & g_{ib} & -2(bG^{-1})^j{}_i \\ -2(bG^{-1})^b{}_a & g_{aj} & g_{ab} & -2(bG^{-1})^j{}_a \\ (G^{-1})^{ib} & 2(G^{-1}b)^i{}_j & 2(G^{-1}b)^i{}_b & (G^{-1})^{ij} \end{pmatrix}. \quad (3.9)$$

3.2 Explicit form of T-duality transformations

Rewriting eq. (3.7) in components we get

$$\begin{aligned} \pm \partial_{\pm} y_a &\cong -2(bG^{-1})^b{}_a \partial_{\pm} y_b + g_{aj} \partial_{\pm} x^j + g_{ab} \partial_{\pm} x^b - 2(bG^{-1})^j{}_a \partial_{\pm} y_j \\ \pm \partial_{\pm} x^i &\cong (G^{-1})^{ib} \partial_{\pm} y_b + 2(G^{-1}b)^i{}_j \partial_{\pm} x^j + 2(G^{-1}b)^i{}_b \partial_{\pm} x^b + (G^{-1})^{ij} \partial_{\pm} y_j \\ \pm \partial_{\pm} x^a &\cong (G^{-1})^{ab} \partial_{\pm} y_b + 2(G^{-1}b)^a{}_j \partial_{\pm} x^j + 2(G^{-1}b)^a{}_b \partial_{\pm} x^b + (G^{-1})^{aj} \partial_{\pm} y_j \\ \pm \partial_{\pm} y_i &\cong -2(bG^{-1})^b{}_i \partial_{\pm} y_b + g_{ij} \partial_{\pm} x^j + g_{ib} \partial_{\pm} x^b - 2(bG^{-1})^j{}_i \partial_{\pm} y_j. \end{aligned} \quad (3.10)$$

Eliminating y_i from the second and third equations we find

$$\Pi_{\mp ab}\partial_{\pm}x^b + \Pi_{\mp ai}\partial_{\pm}x^i + \frac{1}{2}\partial_{\pm}y_a \cong 0. \quad (3.11)$$

Multiplication with $2\kappa\hat{\theta}_{\pm}^{ab}$, which according to (A.10) is the inverse of $\Pi_{\mp ab}$, gives

$$\partial_{\pm}x^a \cong -2\kappa\hat{\theta}_{\pm}^{ab}\Pi_{\mp bi}\partial_{\pm}x^i - \kappa\hat{\theta}_{\pm}^{ab}\partial_{\pm}y_b. \quad (3.12)$$

Similarly, eliminating y_a from the second and third equations we get

$$\Pi_{\mp ij}\partial_{\pm}x^j + \Pi_{\mp ia}\partial_{\pm}x^a + \frac{1}{2}\partial_{\pm}y_i \cong 0, \quad (3.13)$$

which after multiplication with $2\kappa\hat{\theta}_{\pm}^{ij}$, the inverse of $\Pi_{\mp ij}$, produces

$$\partial_{\pm}x^i \cong -2\kappa\hat{\theta}_{\pm}^{ij}\Pi_{\mp ja}\partial_{\pm}x^a - \kappa\hat{\theta}_{\pm}^{ij}\partial_{\pm}y_j. \quad (3.14)$$

The equation (3.12) is the T-duality transformations for x^a (eq. (44) of ref. [9]) and (3.14) is its analogue for x^i .

3.3 T-dual background fields

Requiring that the dual generalized metric (3.9) has the form (2.17) but with T-dual background fields, (denoted by lower index a on the left of background fields)

$${}_a\mathcal{H}_{MN} = \begin{pmatrix} {}_ag^{\mu\nu} & -2({}_ab{}_aG^{-1})^{\mu}{}_{\nu} \\ 2({}_aG^{-1}{}_ab)_{\mu}{}^{\nu} & ({}_aG^{-1})_{\mu\nu} \end{pmatrix}, \quad (3.15)$$

we can find expressions for the T-dual background fields in terms of the initial ones.

It is useful to consider the combination of the dual background fields in the form

$${}_a\Pi_{\pm}^{\mu\nu} \equiv ({}_ab \pm \frac{1}{2}{}_aG)^{\mu\nu} = {}_aG^{\mu\rho}[({}_aG^{-1}{}_ab)_{\rho}{}^{\nu} \pm \frac{1}{2}\delta_{\rho}^{\nu}]. \quad (3.16)$$

Comparing lower D rows of expressions (3.9) and (3.15) we find

$$({}_aG^{-1}{}_ab)_{\mu}{}^{\nu} = \begin{pmatrix} -(bG^{-1})_a{}^b & \frac{1}{2}g_{aj} \\ \frac{1}{2}(G^{-1})^{ib} & (G^{-1}b)^i{}_j \end{pmatrix} \equiv \begin{pmatrix} -\tilde{\beta} & \frac{1}{2}g^T \\ \frac{1}{2}\gamma & -\tilde{\beta}^T \end{pmatrix}, \quad (3.17)$$

and

$$({}_aG^{-1})_{\mu\nu} = \begin{pmatrix} g_{ab} & -2(bG^{-1})_a{}^j \\ 2(G^{-1}b)^i{}_b & (G^{-1})^{ij} \end{pmatrix} \equiv \begin{pmatrix} \tilde{g} & -2\beta_1 \\ -2\beta_1^T & \bar{\gamma} \end{pmatrix}. \quad (3.18)$$

The notation in the second equalities which has been obtained using (A.3), (A.4), (A.6) and (A.7) will simplify calculations.

To obtain background field (3.16) we need the inverse of last expression. We will use the general expression for block wise inversion matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}. \quad (3.19)$$

It produces

$$({}_aG)^{\mu\nu} = \begin{pmatrix} (A^{-1})^{ab} & 2(\tilde{g}^{-1}\beta_1 D^{-1})^a{}_j \\ 2(\tilde{\gamma}^{-1}\beta_1^T A^{-1})_i{}^b & (D^{-1})_{ij} \end{pmatrix}, \quad (3.20)$$

where

$$A_{ab} = (\tilde{g} - 4\beta_1 \tilde{\gamma}^{-1} \beta_1^T)_{ab}, \quad D^{ij} = (\tilde{\gamma} - 4\beta_1^T \tilde{g}^{-1} \beta_1)^{ij}. \quad (3.21)$$

After some direct calculations it can be shown that

$$A_{ab} = (\tilde{G} - 4\tilde{b}\tilde{G}^{-1}\tilde{b})_{ab} \equiv \hat{g}_{ab}, \quad (3.22)$$

where \hat{g}_{ab} has been defined in (A.8). Note that unlike \tilde{g}_{ab} , which is just ab component of $g_{\mu\nu}$, the \hat{g}_{ab} has the same form as effective metric $g_{\mu\nu}$ but with all components (\tilde{G}, \tilde{b}) defined in d dimensional subspace with indices a, b .

Using result (3.22) we can rewrite the first equation (3.21) in the form $\hat{g}_{ab} = \tilde{g}_{ab} - 4(\beta_1 \tilde{\gamma}^{-1} \beta_1^T)_{ab}$. Multiplying it on the left with $(\tilde{g}^{-1})^{ab}$ and on the right with $(\hat{g}^{-1})^{ab}$ we get

$$(\tilde{g}^{-1})^{ab} = (\hat{g}^{-1})^{ab} - 4(\tilde{g}^{-1}\beta_1 \tilde{\gamma}^{-1} \beta_1^T \hat{g}^{-1})^{ab}. \quad (3.23)$$

With the help of this relation we can verify that

$$(D^{-1})_{ij} = (\tilde{\gamma}^{-1} + 4\tilde{\gamma}^{-1}\beta_1^T \hat{g}^{-1} \beta_1 \tilde{\gamma}^{-1})_{ij}, \quad (3.24)$$

is inverse of the second equation (3.21).

Now, we are able to calculate background field (3.16)

$${}_a\Pi_{\pm}^{\mu\nu} = \begin{pmatrix} \tilde{g}^{-1}\beta_1 D^{-1}\gamma - A^{-1}(\tilde{\beta} \mp \frac{1}{2}) & \frac{1}{2}A^{-1}g^T - 2\tilde{g}^{-1}\beta_1 D^{-1}(\tilde{\beta}^T \mp \frac{1}{2}) \\ \frac{1}{2}D^{-1}\gamma - 2\tilde{\gamma}^{-1}\beta_1^T A^{-1}(\tilde{\beta} \mp \frac{1}{2}) & \tilde{\gamma}^{-1}\beta_1^T A^{-1}g^T - D^{-1}(\tilde{\beta}^T \mp \frac{1}{2}) \end{pmatrix}. \quad (3.25)$$

After tedious calculations using (A.5)-(A.7) and (A.11) we can obtain

$${}_a\Pi_{\pm}^{\mu\nu} = \begin{pmatrix} \frac{\kappa}{2}\hat{\theta}_{\mp}^{ab} & \kappa\hat{\theta}_{\mp}^{ab}\Pi_{\pm bi} \\ -\kappa\Pi_{\pm ib}\hat{\theta}_{\mp}^{ba} & \Pi_{\pm ij} - 2\kappa\Pi_{\pm ia}\hat{\theta}_{\mp}^{ab}\Pi_{\pm bj} \end{pmatrix}, \quad (3.26)$$

where $\hat{\theta}_{\pm}^{ab}$ has been defined in (A.9).

It still remains to check that upper D rows of (3.9) and (3.15) produce the same expressions for T-dual background fields. The field $({}_a b {}_a G^{-1})^{\mu}{}_{\nu}$ is just transpose of $({}_a G^{-1} {}_a b)_{\mu}{}^{\nu}$. It is useful to express ${}_a g^{\mu\nu}$ in the form

$${}_a g^{\mu\nu} = ({}_a G)^{\mu\rho} [\delta_{\rho}^{\nu} - 4({}_a G^{-1} {}_a b)_{\rho}{}^{\sigma} ({}_a G^{-1} {}_a b)_{\sigma}{}^{\nu}]. \quad (3.27)$$

Then using (3.20), (3.17), (3.22), and (A.11) we can show that

$${}_a g^{\mu\nu} = \begin{pmatrix} (G^{-1})^{ab} & 2(G^{-1}b)^a{}_j \\ -2(bG^{-1})_i{}^b & g_{ij} \end{pmatrix}, \quad (3.28)$$

which is in agreement with (3.9).

Consequently, we obtained the T-dual background fields in the flat background after dualization along directions x^a , ($a = 0, 1, \dots, d-1$)

$$\begin{aligned} {}_a \Pi_{\pm}^{ab} &= \frac{\kappa}{2} \hat{\theta}_{\mp}^{ab}, & {}_a \Pi_{\pm i}^a &= \kappa \hat{\theta}_{\mp}^{ab} \Pi_{\pm bi}, \\ {}_a \Pi_{\pm i}^a &= -\kappa \Pi_{\pm ib} \hat{\theta}_{\mp}^{ba}, & {}_a \Pi_{\pm ij} &= \Pi_{\pm ij} - 2\kappa \Pi_{\pm ia} \hat{\theta}_{\mp}^{ab} \Pi_{\pm bj}. \end{aligned} \quad (3.29)$$

The symmetric and antisymmetric parts of these expressions produce T-dual metric and T-dual Kalb-Ramond field. This is in complete agreement with the Refs.[9, 24]. The similar way to perform T-duality in the flat space-time for $D = 3$ has been described in App. B of ref. [7].

This proves that exchange the places of some coordinates x^a with its T-dual y_a in the flat double space represents T-dualities along these coordinates.

In Sec. 4.1. of Ref.[3] the Buscher's T-dualities has been derived in eq.(4.9) in the case when there is only one isometry direction. For such a case it was concluded that "the dual background is related to the original one by the action of factorized duality". There is essential difference between such eq.(4.9) and relation (3.29) of the present article, where the general case of T-dualities along arbitrary sets of coordinates has been derived and proof its equivalence with the action of factorized duality.

For proof of expression (3.29) with mathematical induction eq.(4.9) is just first step for $n = 1$. The next step from n to $n + 1$ is nontrivial because then we have three kind of variables (beside isometry one θ there are a set of original variables and a set of variables along which we already performed duality transformations). This leads to the formulae different from eq.(4.9). For example, when we performed T-dualization along more then one coordinate (lets say along x^a , $a = 1, 2$) in expression for T-dual background fields it is not carried out division with G_{aa} as in eq.(4.9) but with $G_{ab} + 2B_{ab}$ which was recorded in expression $\hat{\theta}_{-}^{ab}$ of (3.29).

3.4 T-duality group

Successively T-dualization along disjunct sets of directions \mathcal{T}^{a_1} and \mathcal{T}^{a_2} will produce T-dualization along all directions $a = a_1 \cup a_2$

$$\mathcal{T}^{a_1} \circ \mathcal{T}^{a_2} = \mathcal{T}^a. \quad (3.30)$$

This can be represent by matrix multiplications $(\mathcal{T}^{a_1} \mathcal{T}^{a_2})^M{}_N = (\mathcal{T}^a)^M{}_N$, which is easy to check because the projectors satisfy the relations $P_{a_1}^2 = P_{a_1}$, $P_{a_2}^2 = P_{a_2}$, $P_{a_1} P_{a_2} = 0$ and $P_{a_1} + P_{a_2} = P_a$.

The set of matrices \mathcal{T}^a , where the index a take the values in any of the subset of index μ form a commutative group with respect to matrix multiplication. The neutral element corresponds to the case when we do not perform T-duality, with $P_a = 0$ and $\mathcal{T}^a = 1$. Consequently, the set of all T-duality transformations form a commutative group with respect to the operation \circ .

This is subgroup of the $2D$ permutational group because it acts as a replacement of some coordinates. In two-line notation, the T-duality \mathcal{T}^a , along $2d$ coordinates x^a and y_a can be written as

$$\begin{pmatrix} 1 & 2 & \cdots & d & d+1 & \cdots & D & D+1 & \cdots & D+d & D+d+1 & \cdots & 2D \\ D+1 & D+2 & \cdots & D+d & d+1 & \cdots & D & 1 & \cdots & d & D+d+1 & \cdots & 2D \end{pmatrix}. \quad (3.31)$$

It looks simpler in the cyclic notation

$$(1, D+1)(2, D+2) \cdots (d, D+d). \quad (3.32)$$

We will call this group T-duality group. It is a global symmetry group of equations of motion (2.19).

4 Inclusion of Dilaton field

As usual, in the standard formulation one should add Fradkin-Tseytlin term

$$S_\phi = \int d^2\xi \sqrt{-g} R^{(2)} \phi, \quad (4.1)$$

to the action (2.1) in order to describe dilaton field ϕ . Here $R^{(2)}$ is scalar curvature of the world sheet and the term S_ϕ is one order higher in α' then terms in (2.1).

4.1 Path integral measure

It is well known that dilaton transformation has quantum origin. For a constant background the Gaussian path integral produces the expression $(\det \Pi_{+\mu\nu})^{-1}$. We will show that this is just what we need in order that the change of space-time measure in the path integral is correct.

Let us start with the relations

$$\det G_{\mu\nu} = \frac{\det G_{ab}}{\det \bar{\gamma}^{ij}}, \quad \det {}_a G_{\mu\nu} = \frac{\det {}_a G^{ab}}{\det \bar{\gamma}^{ij}}, \quad (4.2)$$

which follow from (A.1) and (3.18). Using the expressions for T-dual fields (3.29) we can find the relations between the determinants

$$\det(2\Pi_{\pm ab}) = \frac{1}{\det(2{}_a \Pi_{\pm}^{ab})} = \sqrt{\frac{\det G_{\mu\nu}}{\det {}_a G_{\mu\nu}}} = \sqrt{\frac{\det G_{ab}}{\det {}_a G^{ab}}}, \quad (4.3)$$

where the factor 2 introduced for the convenience, because $\Pi_{\pm ab} = B_{ab} \pm \frac{1}{2}G_{ab}$. So, we have

$$\sqrt{\det G_{\mu\nu}} dx^i dx^a \rightarrow \sqrt{\det G_{\mu\nu}} dx^i \frac{1}{\det(2\Pi_{+ab})} dy_a = \sqrt{\det {}_a G_{\mu\nu}} dx^i dy_a, \quad (4.4)$$

which means that T-dualization T^a along x^a directions produces correct change of space-time measure in the path integral of the standard approach.

4.2 Dilaton in the double space

In the double space T-dualization \mathcal{T}^a along both x^a and y_a produces

$$\sqrt{\det G_{\mu\nu}} \sqrt{\det {}^* G^{\mu\nu}} dx^i dy_i dx^a dy_a \rightarrow \quad (4.5)$$

$$\sqrt{\det G_{\mu\nu}} \sqrt{\det {}^* G^{\mu\nu}} dx^i dy_i dy_a dx^a \frac{1}{\det(2\Pi_{+ab}) \det(2{}_a\Pi_+^{ab})}.$$

According to (4.3) the last term is equal to 1 and the path integral measure is invariant under T-dual transformation. Consequently, in double space we need the new dilaton invariant under T-duality transformations.

The usual approach in the literature is to introduce "doubled dilaton" $\Phi^{(a)}$ in term of the standard dilaton ϕ , with requirement that $\Phi^{(a)}$ is invariant under T-dualization \mathcal{T}^a . From the transformation of standard dilaton ϕ (see Refs.[1, 23])

$${}_a\phi = \phi - \ln \det(2\Pi_{+ab}) = \phi - \ln \sqrt{\frac{\det G_{ab}}{\det {}_a G^{ab}}}, \quad (4.6)$$

with the help of (4.3) we have ${}_a({}_a\phi) = \phi$, which means that

$$\Phi^{(a)} = \frac{1}{2}({}_a\phi + \phi) = \phi - \frac{1}{2} \ln \sqrt{\frac{\det G_{ab}}{\det {}_a G^{ab}}}, \quad (4.7)$$

is invariant under duality transformation along x^a directions. If we chose the other set of coordinates x^b ($b \neq a$), along which we perform T-duality, then we wil have different "doubled dilaton" $\Phi^{(b)}$. We want to have one doubled dilaton invariant under all T-duality transformations.

Up to now, we described all T-dual transformations with one action (2.21). Using (2.17) and (2.11) the corresponding generalized metric can be expressed symmetrically in term of initial metric and Kalb-Ramond fields and their totally T-dual background fields (marked with star)

$$\mathcal{H}_{MN} = \begin{pmatrix} ({}^*G^{-1})_{\mu\nu} & 2({}^*G^{-1})^{\mu\rho} {}^*B_{\rho\nu} \\ 2(G^{-1})^{\mu\rho} B_{\rho\nu} & (G^{-1})^{\mu\nu} \end{pmatrix}. \quad (4.8)$$

We can do a similar thing with dilaton field. As was shown in Ref.[10] the expression

$$\Phi = \phi - \ln \sqrt{\det G_{\mu\nu}}, \quad (4.9)$$

is duality invariant under all possible T-dualizations. So, the double space action (2.21) can be extended with the expression similar to (4.1), but with doubled dilaton Φ instead of the standard one ϕ .

Using the fact that $\Phi = {}^* \Phi = {}^* \phi - \ln \sqrt{\det {}^* G^{\mu\nu}}$, we can express double dilaton Φ in term of dilaton from the initial theory ϕ and dilaton from its totally T-dual theory ${}^* \phi$ as

$$e^{-2\Phi} = e^{-(\phi+{}^*\phi)} \sqrt{\det G_{\mu\nu} \det {}^* G^{\mu\nu}}. \quad (4.10)$$

Therefore, we can take $e^{-2\Phi} dx^\mu dy_\mu$ as double space integration measure, as well as in the Double field theory.

5 Relation with the Hull's formulation

In this section we are going to derive the action of Ref.[16] and compare its consequences with our results. Note that the background fields of Ref.[16] depend only on the coordinates Y^m (x^i in our notation) along which the T-duality has not been executed. In our approach all variables are doubled $x^\mu \rightarrow y_\mu$, while in Ref.[16] only variables along which the T-duality is performed are doubled $x^a \rightarrow y_a$. So, in our approach there are $2D$ variables x^a, x^i, y_a, y_i while in Ref.[16] there are $D+d$ variables x^a, y_a, x^i . It suggest that formulation of Ref.[16] can be obtained from our one after elimination of y_i variable. We already did it in Subsec. 3.2. and obtain T-duality relations (3.11) and (3.12), which are inverse to each other. In analogy with (2.13) we can rewrite them in a useful form, where on the left hand side we put the terms with world-sheet antisymmetric tensor ε_α^β ($\varepsilon_\pm^\pm = \pm 1$) and obtain

$$\partial_\pm X^A = \pm \Omega^{AB} (\hat{\mathcal{H}}_{BC} \partial_\pm X^C + J_{\pm B}). \quad (5.11)$$

Here

$$X^A = \begin{pmatrix} x^a \\ y_a \end{pmatrix}, \quad (5.12)$$

is $2d$ dimensions double space coordinate

$$\Omega^{AB} = \begin{pmatrix} 0 & 1_a \\ 1_a & 0 \end{pmatrix}, \quad (5.13)$$

and

$$\hat{\mathcal{H}}_{AB} = \begin{pmatrix} \hat{g}_{ab} & -2b_{ac}(\tilde{G}^{-1})^{cb} \\ 2(\tilde{G}^{-1})^{ac}b_{cb} & (\tilde{G}^{-1})^{ab} \end{pmatrix}, \quad (5.14)$$

is $2d \times 2d$ generalized metric. All components of $\hat{\mathcal{H}}_{AB}$ are from ab subspace, like \hat{g}_{ab} and $\hat{\theta}^{ab}$ in (A.8). So, it satisfies $\hat{\mathcal{H}}^T \Omega \hat{\mathcal{H}} = \Omega$ and it is element of $O(d, d)$ group.

We also obtained the explicit expressions for the currents in terms of undualized coordinates x^i

$$J_{\pm A} = \begin{pmatrix} J_{1\pm a} \\ J_{2\pm}^a \end{pmatrix}, \quad (5.15)$$

where

$$J_{1\pm a} = -2\Pi_{\pm ab} J_{2\pm}^b, \quad J_{2\pm}^a = 2(\tilde{G}^{-1})^{ab} \Pi_{\mp bi} \partial_{\pm} x^i. \quad (5.16)$$

The first relation in the last expression is solution (2.44) of Ref.[16].

Therefore, instead of $2D$ component T-duality transformations (3.7) with $2D$ dimensional vector Z^M we have $2d$ component relation (5.11) with $2d$ dimensional vectors X^A and $J_{\pm A}$. The relation (5.11) is self-duality constraints (eq.(2.5) of Ref.[16]) imposed that halves the degrees of freedom.

As well as in Subsec. 2.3 consistency condition of (5.11) produces

$$\partial_+(\hat{\mathcal{H}}\partial_-X + J_-) + \partial_-(\hat{\mathcal{H}}\partial_+X + J_+) = 0, \quad (5.17)$$

which is equation of motion (2.4) of Ref.[16]. So, we can write the action

$$S_d = \frac{\kappa}{4} \int d^2\xi \left[\partial_+ X^A \hat{\mathcal{H}}_{AB} \partial_- X^B + \partial_+ X^A J_{-A} + J_{+A} \partial_- X^A + \mathcal{L}(x^i) \right], \quad (5.18)$$

which variation produce the eq.(5.17). This action is in complete agreement with Ref.[16], but with already constrained elements of $\hat{\mathcal{H}}_{AB}$ and explicit expression for currents $J_{\pm A}$ in terms of undualized coordinates x^i . Because Ref.[16] starts with arbitrary $\hat{\mathcal{H}}_{AB}$ it is restricted to be coset metric $O(d, d)/O(d) \times O(d)$ that the T-duality would be equivalent to the standard one.

Note that the whole procedure of the Ref.[16] should be performed for each node of the chain (1.1), which means for each values of d . In our approach only the case $d = D$ appears. Then the currents $J_{\pm A}$ disappear, $X^A \rightarrow Z^M$, $\hat{\mathcal{H}}_{AB} \rightarrow \mathcal{H}_{MN}$, $\Omega^{AB} \rightarrow \Omega^{MN}$ and T-duality transformations (5.11) turns to (2.15). However, the generalized metric \mathcal{H}_{MN} together with basic relation (2.15) are sufficient to describe all T-dualities for each d .

6 Conclusion

Introducing the $2D$ dimensional space, which beside initial D dimensional space-time coordinates x^μ contains the corresponding T-dual coordinates y_μ , we offered simple formulation for T-duality transformations. The extended space with the coordinates $Z^M = (x^\mu, y_\mu)$ we call double space.

It is easy to see that after the exchanges of all initial coordinates x^μ with all T-dual coordinates y_μ we obtain the T-dual background fields of Sec.2. This result is formulated

in the double space in Ref.[11] in order to make global $SO(D, D)$ symmetry manifest. In the present article we show that the double space contains enough information to explain T-dualization along arbitrary subset of coordinates x^a and corresponding T-dual y_a ($a = 0, 1, \dots, d-1$). For this purpose we rewrite T-duality transformations for all the coordinates and its inverse in the double space. We obtain the basic relation (2.15) with the generalized metric (2.17) which help us to find all T-dual background fields for each node of the chain (1.1) and T-duality transformations between all the nodes.

We define particular permutation of the coordinates realized by matrix \mathcal{T}^a , known in literature as factorized duality (see for example [3]). It exchanges the places of some subset of the coordinates x^a and the corresponding dual coordinates y_a along which we perform T-dualization. We require that the obtained double space coordinates satisfy the same form of T-duality transformations as the initial one, or in other words that such permutation is a global symmetry of the T-dual transformation. We show that this permutation produce exactly the same T-dual background fields and T-duality transformations as in the standard approach of Ref.[9]. So, double space approach clearly explains that T-duality is nonphysical, because it is equivalent to the permutation of some coordinates in the double space.

In the standard formulation T-duality transforms the initial theory to the equivalent one, T-dual theory. The double space formulation contains both initial and T-dual theories and T-duality becomes the global symmetry transformation. With the help of (3.8) it is, easy to see that equations of motion (2.19) are invariant under transformation $Z^M \rightarrow Z_a^M = (\mathcal{T}^a)^M_N Z^N$.

The square of all matrix \mathcal{T}^a are equal to one and therefore they are inverse themselves. The set of all \mathcal{T}^a matrices form an Abelian group with respect to the matrix multiplication. Consequently, the set of all T-dualizations with respect to the successive T-dualizations also form an Abelian group. It is a subgroup of the $2D$ permutation group, which permute some of the first D coordinates with corresponding last D coordinates. In the cyclic form it can be written as

$$(1, D+1)(2, D+2) \cdots (d, D+d), \quad d = 0, 1, 2, \dots, D, \quad (6.19)$$

where $d = 0$ formally corresponds to the neutral element (no permutations of coordinates and so no T-duality transformations) and $d = D$ corresponds to the case when T-dualization is performed along all coordinates.

The relation between our approach and the well known one of Ref.[16] has been presented in Sec.4. In approach of Ref.[16] to each node of the chain (1.1), lying d steps from the beginning, it corresponds the action S_d (5.18) and self-duality constraints (5.11) with $2d$ dimensional variables X^A . Our approach unify all nodes of the chain (1.1). The T-duality transformations (2.15), with $2D$ dimensional variable Z^M , allows us to obtain all background fields and T-duality transformations of the chain (1.1).

A Block-wise expressions for background fields

In order to simplify notation and to write expressions without indices (as matrix multiplication) we will introduce notations for component fields.

For the metric tensor and the Kalb-Ramond background fields we define

$$G_{\mu\nu} = \begin{pmatrix} \tilde{G}_{ab} & G_{aj} \\ G_{ib} & \bar{G}_{ij} \end{pmatrix} \equiv \begin{pmatrix} \tilde{G} & G^T \\ G & \bar{G} \end{pmatrix}, \quad (\text{A.1})$$

and

$$b_{\mu\nu} = \begin{pmatrix} \tilde{b}_{ab} & b_{aj} \\ b_{ib} & \bar{b}_{ij} \end{pmatrix} \equiv \begin{pmatrix} \tilde{b} & -b^T \\ b & \bar{b} \end{pmatrix}. \quad (\text{A.2})$$

We also define notation for inverse of the matrix

$$(G^{-1})^{\mu\nu} = \begin{pmatrix} \tilde{\gamma}^{ab} & \gamma^{aj} \\ \gamma^{ib} & \bar{\gamma}^{ij} \end{pmatrix} \equiv \begin{pmatrix} \tilde{\gamma} & \gamma^T \\ \gamma & \bar{\gamma} \end{pmatrix}, \quad (\text{A.3})$$

and for the effective metric

$$g_{\mu\nu} = G_{\mu\nu} - 4b_{\mu\rho}(G^{-1})^{\rho\sigma}b_{\sigma\nu} = \begin{pmatrix} \tilde{g}_{ab} & g_{aj} \\ g_{ib} & \bar{g}_{ij} \end{pmatrix} \equiv \begin{pmatrix} \tilde{g} & g^T \\ g & \bar{g} \end{pmatrix}. \quad (\text{A.4})$$

Note that because $G^{\mu\nu}$ is inverse of $G_{\mu\nu}$ we have

$$\begin{aligned} \gamma &= -\bar{G}^{-1}G\tilde{\gamma} = -\tilde{\gamma}G\tilde{G}^{-1}, & \gamma^T &= -\tilde{G}^{-1}G^T\tilde{\gamma} = -\tilde{\gamma}G^T\bar{G}^{-1}, \\ \tilde{\gamma} &= (\tilde{G} - G^T\bar{G}^{-1}G)^{-1}, & \bar{\gamma} &= (\bar{G} - G\tilde{G}^{-1}G^T)^{-1}, \\ \tilde{G}^{-1} &= \tilde{\gamma} - \gamma^T\bar{\gamma}^{-1}\gamma, & \bar{G}^{-1} &= \bar{\gamma} - \gamma\tilde{\gamma}^{-1}\gamma^T. \end{aligned} \quad (\text{A.5})$$

It is also useful to introduce new notation for expressions

$$(bG^{-1})_\mu{}^\nu = \begin{pmatrix} \tilde{b}\tilde{\gamma} - b^T\gamma & \tilde{b}\gamma^T - b^T\bar{\gamma} \\ b\tilde{\gamma} + \bar{b}\gamma & b\gamma^T + \bar{b}\bar{\gamma} \end{pmatrix} \equiv \begin{pmatrix} \tilde{\beta} & \beta_1 \\ \beta_2 & \bar{\beta} \end{pmatrix}, \quad (\text{A.6})$$

and

$$(G^{-1}b)^\mu{}_\nu = \begin{pmatrix} \tilde{\gamma}\tilde{b} + \gamma^Tb & -\tilde{\gamma}b^T + \gamma^T\bar{b} \\ \gamma\tilde{b} + \bar{\gamma}b & -\gamma b^T + \bar{\gamma}\bar{b} \end{pmatrix} \equiv \begin{pmatrix} -\tilde{\beta}^T & -\beta_2^T \\ -\beta_1^T & -\bar{\beta}^T \end{pmatrix}. \quad (\text{A.7})$$

We denote by $\hat{}$ expressions similar to the effective metric (A.4) and non-commutativity parameters but with all contributions from ab subspace

$$\hat{g}_{ab} = (\tilde{G} - 4\tilde{b}\tilde{G}^{-1}\tilde{b})_{ab}, \quad \hat{\theta}^{ab} = -\frac{2}{\kappa}(\hat{g}^{-1}\tilde{b}\tilde{G}^{-1})^{ab}. \quad (\text{A.8})$$

Note that $\hat{g}_{ab} \neq \tilde{g}_{ab}$ because \tilde{g}_{ab} is projection of $g_{\mu\nu}$ on subspace ab . It is extremely useful to introduce background field combinations

$$\Pi_{\pm ab} = b_{ab} \pm \frac{1}{2}G_{ab} \quad \hat{\theta}_{\pm}^{ab} = -\frac{2}{\kappa}(\hat{g}^{-1}\tilde{\Pi}_{\pm}\tilde{G}^{-1})^{ab} = \hat{\theta}^{ab} \mp \frac{1}{\kappa}(\hat{g}^{-1})^{ab}, \quad (\text{A.9})$$

which are inverse to each other

$$\hat{\theta}_{\pm}^{ac}\Pi_{\mp cb} = \frac{1}{2\kappa}\delta_b^a. \quad (\text{A.10})$$

With the help of (3.23) one can prove the relation

$$(\tilde{g}^{-1}\beta_1 D^{-1})^a{}_i = (\hat{g}^{-1}\beta_1 \tilde{\gamma}^{-1})^a{}_i, \quad (\text{A.11})$$

where D^{ij} is defined in (3.21).

References

- [1] T. Buscher, *Phys. Lett.* **B 194** (1987) 59; **201** (1988) 466.
- [2] M. Roček and E. Verlinde, *Nucl.Phys.* **B 373** (1992) 630.
- [3] A. Giveon, M.Porrati and E. Rabinovici, *Phys. Rep.* **244** (1994) 77.
- [4] E. Alvarez, L. Alvarez-Gaume, J. Barbon and Y. Lozano, *Nuc. Phys.* **B 415** (1994) 71.
- [5] Lj. Davidović and B. Sazdović, *EPJ C* **74** (2014) 2683.
- [6] D. Lust *JHEP* **12** (2010) 084.
- [7] D. Andriot, M. Larfors, D. Lust and P. Patalong, *JHEP* **06** (2013) 021.
- [8] Lj. Davidović, B. Nikolić and B. Sazdović, *EPJ C* **74** (2014) 2734.
- [9] Lj. Davidović, B. Nikolić and B. Sazdović, arXiv: 1406.5364.
- [10] B. Sazdović, *JHEP* **08** (2015) 055.
- [11] M. Duff, *Nucl. Phys.* **B 335** (1990) 610.
- [12] A. A. Tseytlin, *Phys.Lett.* **B 242** (1990) 163.
- [13] A. A. Tseytlin, *Nucl. Phys.* **B 350** (1991) 395.
- [14] W. Siegel, *Phys.Rev.* **D 48** (1993) 2826.
- [15] W. Siegel, *Phys.Rev.* **D 47** (1993) 5453.
- [16] C. M. Hull, *JHEP* **10** (2005) 065.
- [17] C. M. Hull, *JHEP* **10** (2007) 057; **07** (2007) 080.

- [18] K. Becker, M. Becker and J. Schwarz *String Theory and M-Theory: A Modern Introduction*; B. Zwiebach, *A First Course in String Theory*, Cambridge University Press, 2004.
- [19] C. Hull, B. Zwiebach, *JHEP* **09** (2009) 099; *JHEP* **09** (2009) 090.
- [20] O. Hohm, C. Hull, B. Zwiebach, *JHEP* **08** (2010) 008.
- [21] G. Aldazabal, D. Marques, and C. Nunez, *Class.Quant.Grav.* **30** (2013) 163001.
- [22] D. S. Berman and D. C. Thompson, *Duality Symmetric String and M-Theory*, arXiv:1306.2643.
- [23] A. Giveon and M. Roček, *Nucl. Phys.* **B 421** (1994) 173.
- [24] B. Nikolić and B. Sazdović, *Nucl. Phys.* **B 836** (2010) 100.